2019 USAPhO Question B3 (c) Lagrangian Mechanics Approach *

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1. My Motivation of Writing This Solution

Although this is the last part of the last problem in 2019 USAPhO contest, it can be easily solved by using the conservation of the system's horizontal-component momentum and the conservation of the system's total energy.

However, AAPT's official solution has the following comment that causes my surprise: *"Another common route was to apply Lagrangian mechanics, solving the Euler-Lagrange equations, or equivalently to solve the $F = ma$ equations.* ***These are quite complicated, and nobody managed to integrate them to get the correct answer***[1]*."*

My original intuition about why the Lagrangian mechanics approach should not be too hard is as follows. If we use Lagrangian mechanics to solve this problem, we only need to introduce two generalized coordinates: the position of the bead and the angle of the rod. Each object's kinetic energy and gravitational potential energy are also simple functions of these two generalized coordinates. So the two Euler-Lagrange equations can be easily derived and are not too complicated. Based on my intuition, I did not understand why no student who used this approach successfully solved these seemingly simple equations.

With this surprise and curiosity why ALL students who took this approach were stuck in the contest, I decided to use the Lagrangian mechanics approach to fully solve this problem. I aim to achieve the following objectives from my solution in this note.

1. From a Physics Olympiad coach's perspective, I need to diagnose where students were stuck. Therefore, I can get a clear sense what specific problem-solving skills I shall teach my future students to help them get through this type of problems.

2. I must admit that Lagrangian mechanics is not a quick and simple approach to this problem. I would not recommend students to use this approach if they saw a similar problem. However, it does not entail that Lagrangian mechanics approach is not useful. Nobody knows whether on some day, AAPT will design a USAPhO problem that the easiest way to solve it is to use Lagrangian mechanics, not Newtonian mechanics. Therefore, I want to use this note to raise student awareness of the existence of this non-Newtonian mechanics approach and its power to solve problems.

[1] https://www.aapt.org/Common/upload/USAPhO-2019-Solutions_rev-4012021.pdf

The rest of this note is organized as follows. In §2, I recap the problem. In §3, I present my solution. In §4, I summarize some skills that I used to solve this problem.

2. Problem

Pitfall

A bead is placed on a horizontal rail, along which it can slide frictionlessly. It is attached to the end of a rigid, massless rod of length R. A ball is attached at the other end. Both the bead and the ball have mass M. The system is initially stationary, with the ball directly above the bead. The ball is then given an infinitesimal push, *parallel* to the rail.

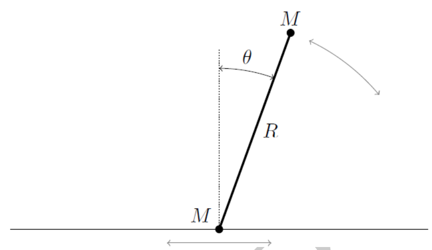

Figure 1: 2019 USAPhO Question B3, Figure 1

Assume that the rod and ball are designed in such a way (not shown explicitly in the diagram) so that they can pass through the rail without hitting it. In other words, the rail only constrains the motion of the bead. Two subsequent states of the system are shown below.

Figure 2: 2019 USAPhO Question B3, Figure 2

(a) Derive an expression for the force in the rod when it is horizontal, as shown at left above, and indicate whether it is tension or compression.

(b) Derive an expression for the force in the rod when the ball is directly below the bead, as shown at right above, and indicate whether it is tension or compression.

(c) Let θ be the angle the rod makes with the vertical, so that the rod begins at $\theta = 0$. Find the angular velocity $\omega = d\theta/dt$ as a function of θ.

3. My Solution

I present my solution of using the Lagrangian mechanics approach in this section. Recall that this problem has three parts. However, this note only focuses on Part c. So we skip solving Parts a and b.

We establish a coordinate system in the diagram. We set the bead's initial position as the origin. We put the x-axis on the horizontal rail. The positive direction is toward the right. The unit vector is denoted as $\hat{\boldsymbol{i}}$. We put the y-axis in the upright position. The positive direction is toward the top. The unit vector is denoted as $\hat{\boldsymbol{j}}$.

We define two generalized coordinates. One is the bead's x-component coordinate, x. The other one is the rod's angle from the upright position, θ.

It is worth noting that these generalized coordinates are functions of time t. To lighten notation, we suppress their dependencies on t unless necessary.

Thus, the position of the bead is $\overrightarrow{\boldsymbol{r_1}} = x\ \hat{\boldsymbol{i}}$. The position of the ball is $\overrightarrow{\boldsymbol{r_2}} = (x + R\sin\theta)\ \hat{\boldsymbol{i}} + R\cos\theta\ \hat{\boldsymbol{j}}$.

Hence, the velocity of the bead is given by

$$\begin{aligned}\overrightarrow{\boldsymbol{v_1}} &= \frac{d\overrightarrow{\boldsymbol{r_1}}}{dt} \\ &= \dot{x}\ \hat{\boldsymbol{i}}.\end{aligned}$$

The velocity of the ball is given by

$$\begin{aligned}\overrightarrow{\boldsymbol{v_2}} &= \frac{d\overrightarrow{\boldsymbol{r_2}}}{dt} \\ &= \left(\dot{x} + R\cos\theta\cdot\dot{\theta}\right)\ \hat{\boldsymbol{i}} - R\sin\theta\cdot\dot{\theta}\ \hat{\boldsymbol{j}}\end{aligned}$$

Therefore, the system's total kinetic energy is

$$T = \frac{1}{2}Mv_1^2 + \frac{1}{2}Mv_2^2.$$

The system's total potential energy is

$$V = MgR\cos\theta.$$

Hence, the system's Lagrangian is

$$\begin{aligned}\mathcal{L} &= T - V \\ &= \frac{1}{2}M\left(2\dot{x}^2 + R^2\dot{\theta}^2 + 2\dot{x}R\cos\theta\cdot\dot{\theta}\right) - MgR\cos\theta. \qquad (1)\end{aligned}$$

We have the following Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{\partial\mathcal{L}}{\partial x} \qquad (2)$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\partial\mathcal{L}}{\partial\theta} \qquad (3)$$

Plugging (1) into (2) and (3), we get

$$\frac{d}{dt}\left(2\dot{x} + R\cos\theta\cdot\dot{\theta}\right) = 0 \qquad (2')$$

$$\frac{d}{dt}\left(R\dot{\theta} + \dot{x}\cos\theta\right) = -\dot{x}\sin\theta\cdot\dot{\theta} + g\sin\theta \qquad (3')$$

Now, we solve equations (2') and (3').

Taking integral of (2'), we get

$$2\dot{x} + R\cos\theta \cdot \dot{\theta} = \text{const.}$$

We have the initial values $x_0 = 0$, $\theta_0 = 0$, $\dot{\theta}_0 = 0$. Thus, the constant above is 0. Thus,

$$2\dot{x} + R\cos\theta \cdot \dot{\theta} = 0. \qquad (2'')$$

Plugging (2") into (3') to eliminate $\dot{x}$, we get the following equation:

$$R\left(1+\sin^2\theta\right)\ddot{\theta} + R\sin\theta\cos\theta \cdot \dot{\theta}^2 - 2g\sin\theta = 0. \qquad (4)$$

Next, we solve this equation. We have

$$\begin{aligned}\ddot{\theta} &= \frac{d\dot{\theta}}{dt} \\ &= \frac{d\dot{\theta}}{d\theta} \cdot \frac{d\theta}{dt} \\ &= \frac{d\dot{\theta}}{d\theta}\dot{\theta}. \qquad (5)\end{aligned}$$

Plugging (5) into (4), we get

$$R\left(1+\sin^2\theta\right)\frac{d\dot{\theta}}{d\theta}\dot{\theta} + R\sin\theta\cos\theta \cdot \dot{\theta}^2 - 2g\sin\theta = 0.(4')$$

Multiplying both sides of (4') by $d\theta$, then the R.H.S. of (4') is still 0 and the L.H.S. of (4') can be reorganized as follows:

$$\begin{aligned}&R\left(1+\sin^2\theta\right)d\dot{\theta}\cdot\dot{\theta} + R\sin\theta\cos\theta\cdot\dot{\theta}^2 d\theta - 2g\sin\theta d\theta \\ &= \frac{R}{2}\left(1+\sin^2\theta\right)d\dot{\theta}^2 + R\sin\theta\cos\theta\cdot\dot{\theta}^2 d\theta - 2g\sin\theta d\theta \\ &= \frac{R}{2}\left(1+\sin^2\theta\right)d\dot{\theta}^2 + \frac{R}{2}\sin 2\theta\cdot\dot{\theta}^2 d\theta - 2g\sin\theta d\theta \\ &= \frac{R}{2}\left(1+\sin^2\theta\right)d\dot{\theta}^2 - \frac{R}{4}\dot{\theta}^2 d\cos 2\theta + 2gd\cos\theta \\ &= \frac{R}{2}\left(1+\sin^2\theta\right)d\dot{\theta}^2 - \frac{R}{4}\left(d\left(\dot{\theta}^2\cos 2\theta\right) - \cos 2\theta d\dot{\theta}^2\right) + 2gd\cos\theta \\ &= \frac{R}{4}\left(2+2\sin^2\theta + \cos 2\theta\right)d\dot{\theta}^2 - \frac{R}{4}d\left(\dot{\theta}^2\cos 2\theta\right) + 2gd\cos\theta \\ &= \frac{3R}{4}d\dot{\theta}^2 - \frac{R}{4}d\left(\dot{\theta}^2\cos 2\theta\right) + 2gd\cos\theta \\ &= d\left(\frac{3R}{4}\dot{\theta}^2 - \frac{R}{4}\dot{\theta}^2\cos 2\theta + 2g\cos\theta\right).\end{aligned}$$

The first equality follows from the property that $udu = \frac{1}{2}du^2$. The second equality follows from the property that $2\sin\theta\cos\theta = \sin 2\theta$. The third equality follows from the property that $\sin udu = -d\cos u$. The fourth equality follows from the property that $udv = d(uv) - vdu$. The sixth equality follows from the property that $\cos 2\theta = 1 - 2\sin^2\theta$.

Because the above quantity is equal to 0, taking the integral, we get

$$\frac{3R}{4}\dot{\theta}^2 - \frac{R}{4}\dot{\theta}^2 \cos 2\theta + 2g\cos\theta = \text{const}.$$

We have the initial values $\theta_0 = 0$, $\dot{\theta}_0 = 0$. Thus, the constant above is $2g$.
Therefore,

$$\frac{3R}{4}\dot{\theta}^2 - \frac{R}{4}\dot{\theta}^2 \cos 2\theta + 2g\cos\theta = 2g.$$

Therefore,

$$\begin{aligned}\dot{\theta}^2 &= \frac{8g}{R}\frac{1-\cos\theta}{3-\cos 2\theta} \\ &= \boxed{\frac{4g}{R}\frac{1-\cos\theta}{1+\sin^2\theta}}.\end{aligned}$$

The second equality follows from the property that $\cos 2\theta = 1 - 2\sin^2\theta$.

4. Some Problem-Solving Skills that I Used

It is worth highlighting some problem-solving skills that I used to solve this problem. I realize not lots of students are familiar with these skills. So it is valuable for them to learn these skills here.

1. **(Try to avoid generating 2nd-order derivative terms)** In general, it is harder to solve differential equations with 2nd-order derivative terms than those that have up to the 1st-order derivative terms.

   For example, for (2'), because the R.H.S. is 0, we can immediately see that the term within $\frac{d}{dt}(\cdot)$ on the L.H.S. is a constant. Doing so gives us Equation (2") that has up to the 1st-order derivative terms only.

   By contrast, for (2'), if we take the derivative, then we get an equation with 2nd-order derivative terms. This complicates the subsequent analysis or even makes us impossible to solve the problem.

2. **(Express a 2nd-order derivative in terms of its 0th and 1st-order derivatives)**

   We are not always as lucky as Equation (2') that can be converted to a form without having any 2nd-order derivative term. For example, Equation (4) has a 2nd-order derivative term $\ddot{\theta}$. In such a situation, we can apply the skill that I used in §3 to make the following conversion:

   $$\ddot{\theta} = \frac{d\dot{\theta}}{d\theta}\dot{\theta}. \qquad (5)$$

   After this manipulation, our equation no longer has $\ddot{\theta}$. By contrast, this 2nd-order derivative term is replace by some operations and combinations of the 0th-order term θ and the 1st-order term $\dot{\theta}$. Therefore, in the rest of the analysis, we only need to solve an equation with θ and $\dot{\theta}$, not $\ddot{\theta}$.

To conclude this note, we show another application of Equation (5). We use it to establish a very crucial result in Hamiltonian mechanics, another useful scheme to characterize physics systems.

Theorem 1. *Consider the following harmonic oscillator of mass m:*

$$\ddot{x} + \omega^2 x = 0,$$

where x is the position. Denote by p the momentum.

Then in the phase space, the trajectory of (x, p) is an ellipse.

Proof. Applying Equations (5) into the harmonic equation in this theorem, we get

$$\frac{d\dot{x}}{dx} \cdot \dot{x} + \omega^2 x = 0.$$

Thus,

$$\dot{x} d\dot{x} + \omega^2 x dx = 0.$$

Thus,

$$d\frac{\dot{x}^2}{2} + \omega^2 d\frac{x^2}{2} = 0.$$

Multiplying both sides by $2m^2$ and applying the fact that $p = m\dot{x}$, we get

$$d\left(p^2 + m^2\omega^2 x^2\right) = 0.$$

Hence,

$$p^2 + m^2\omega^2 x^2 = C,$$

where the constant C can be computed with the initial values of x and p.

Therefore, in the phase space, the trajectory of (x, p) is an ellipse. ■