2023 AMC 10B
Problems and Solutions *

PROFESSOR CHEN EDUCATION PALACE

1. (2023 AMC 10B Problem 1)(2023 AMC 12B Problem 1)

   Mrs. Jones is pouring orange juice into four identical glasses for her four sons. She fills the first three glasses completely but runs out of juice when the fourth glass is only $\frac{1}{3}$ full. What fraction of a glass must Mrs. Jones pour from each of the first three glasses into the fourth glass so that all four glasses will have the same amount of juice?

   A. $\frac{1}{12}$ B. $\frac{1}{4}$ C. $\frac{1}{6}$ D. $\frac{1}{8}$ E. $\frac{2}{9}$

   Solution: $\boxed{\textbf{(C) } \frac{1}{6}}$.

2. (2023 AMC 10B Problem 2)(2023 AMC 12B Problem 2)

   Carlos went to a sports store to buy running shoes. Running shoes were on sale, with prices reduced by 20% on every pair of shoes. Carlos also knew that he had to pay a 7.5% sales tax on the discounted price. He had \$43. What is the original (before discount) price of the most expensive shoes he could afford to buy?

   A. \$46 B. \$50 C. \$48 D. \$47 E. \$49

   Solution: $\boxed{\textbf{(B) \$50}}$.

3. (2023 AMC 10B Problem 3)(2023 AMC 12B Problem 3)

   A 3-4-5 right triangle is inscribed in circle A, and a 5-12-13 right triangle is inscribed in circle B. What is the ratio of the area of circle A to the area of circle B?

   A. $\frac{9}{25}$ B. $\frac{1}{9}$ C. $\frac{1}{5}$ D. $\frac{25}{169}$ E. $\frac{4}{25}$

   Solution: $\boxed{\textbf{(D) } \frac{25}{169}}$.

4. (2023 AMC 10B Problem 4)(2023 AMC 12B Problem 4)

   Jackson's paintbrush makes a narrow strip with a width of 6.5 millimeters. Jackson has enough paint to make a strip 25 meters long. How many square centimeters of paper could Jackson cover with paint?

   A. 162,500 B. 162.5 C. 1,625 D. 1,625,000 E. 16,250

   Solution: **(C) 1,625**

5. (2023 AMC 10B Problem 5)

   Maddy and Lara see a list of numbers written on a blackboard. Maddy adds 3 to each number in the list and finds that the sum of her new numbers is 45. Lara multiplies each number in the list by 3 and finds that the sum of her new numbers is also 45. How many numbers are written on the blackboard?

   A. 10 B. 5 C. 8 D. 6 E. 9

   Solution: Denote by n the numbers written on the blackboard. Denote by S the sum of these numbers.

   We have $S + 3n = 45$ and $3S = 45$. By solving these equations, we get $n =$ **(A) 10**.

6. (2023 AMC 10B Problem 6)

   Let $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$. How many terms in the sequence $L_1, L_2, L_3, \cdots, L_{2023}$ are even?

   A. 673 B. 1011 C. 675 D. 1010 E. 674

   Solution: The parity of this sequence has period 3. In each period, the parity pattern is: odd, odd, even. We have $2023 = 3 \cdot 674 + 1$. Therefore, the number of terms in the given sequence that are even is **(E) 674**.

7. (2023 AMC 10B Problem 7)

   Square $ABCD$ is rotated 20° clockwise about its center to obtain square $EFGH$, as shown below. What is the degree measure of $\angle EAB$?

   A. 24° B. 35° C. 30° D. 32° E. 20°

   Solution: Denote by O the center of two squares. We have $OA = OE$ and $\angle AOE = 20^\circ$. Thus, $\angle EAO = \frac{180^\circ - \angle AOE}{2} = 80^\circ$.

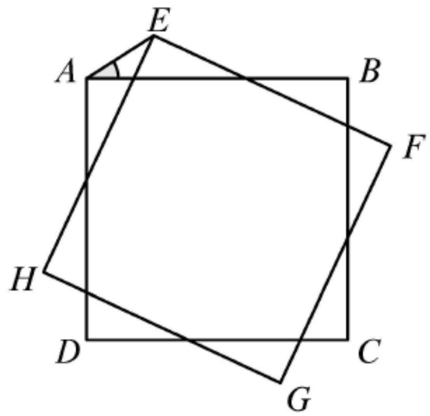

Figure 1: 2023 AMC 10B Problem 7

We have

$$\begin{aligned}\angle EAB &= \angle EAO - \angle BAO \\ &= 80^\circ - 45^\circ \\ &= \boxed{\textbf{(B) } 35^\circ}.\end{aligned}$$

8. (2023 AMC 10B Problem 8)

   What is the units digit of $2022^{2023} + 2023^{2022}$?

   A. 7 B. 1 C. 9 D. 5 E. 3

Solution: Module 10, we have

$$\begin{aligned}2022^{2023} + 2023^{2022} &\equiv 2^{2023} + 3^{2022} \\ &\equiv 2^3 + 3^2 \\ &\equiv 8 + 9 \\ &\equiv \boxed{\textbf{(A) 7}}.\end{aligned}$$

9. (2023 AMC 10B Problem 9)

   The numbers 16 and 25 are a pair of consecutive positive perfect squares whose difference is 9. How many pairs of consecutive positive perfect squares have a difference of less than or equal to 2023?

   A. 674 B. 1011 C. 1010 D. 2019 E. 2017

Solution: Consider a pair of consecutive perfect squares x^2 and $(x+1)^2$. The difference is

$$(x+1)^2 - x^2 = 2x + 1.$$

To get this difference less than or equal to 2023, we need to have

$$2x + 1 \leq 2023.$$

Thus, $x \leq 1011$. Because $x \geq 1$, the number of solutions is **(B) 1011**.

10. (2023 AMC 10B Problem 10)(2023 AMC 12B Problem 5)

You are playing a game. A 2×1 rectangle covers two adjacent squares (oriented either horizontally or vertically) of a 3×3 grid of squares, but you are not told which two squares are covered. Your goal is to find at least one square that is covered by the rectangle. A "turn" consists of you guessing a square, after which you are told whether that square is covered by the hidden rectangle. What is the minimum number of turns you need to ensure that at least one of your guessed squares is covered by the rectangle?

A. 3 B. 5 C. 4 D. 8 E. 6

Solution: **(C) 4**.

11. (2023 AMC 10B Problem 11)

Suzanne went to the bank and withdrew \$800. The teller gave her this amount using \$20 bills, \$50 bills, and \$100 bills, with at least one of each denomination. How many different collections of bills could Suzanne have received?

A. 45 B. 21 C. 36 D. 28 E. 32

Solution: Denote by x, y, z the amount of \$20 bills, \$50 bills and \$100 bills, respectively. Thus, we need to find the number of tuples (x, y, z) with $x, y, z \in \mathbb{N}$ that satisfy

$$20x + 50y + 100z = 800.$$

First, this equation can be simplified as

$$2x + 5y + 10z = 80.$$

Second, we must have $5|x$. Denote $x = 5x'$. The above equation can be converted to

$$2x' + y + 2z = 16.$$

Third, we must have $2|y$. Denote $y = 2y'$. The above equation can be converted to

$$x' + y' + z = 8.$$

Denote $x'' = x' - 1$, $y'' = y' - 1$ and $z'' = z - 1$. Thus, the above equation can be written as

$$x'' + y'' + z'' = 5.$$

Therefore, the number of non-negative integer solutions (x'', y'', z'') is $\binom{5+3-1}{3-1} = \boxed{\textbf{(B) } 21}$.

12. (2023 AMC 10B Problem 12)(2023 AMC 12B Problem 6)

When the roots of the polynomial

$$P(x) = (x-1)^1(x-2)^2(x-3)^3 \cdots (x-10)^{10}$$

are removed from the real number line, what remains is the union of 11 disjoint open intervals. On how many of these intervals is $P(x)$ positive?

A. 3 B. 7 C. 6 D. 4 E. 5

Solution: $\boxed{\textbf{(C) } 6}$.

13. (2023 AMC 10B Problem 13)(2023 AMC 12B Problem 9)

What is the area of the region in the coordinate plane defined by

$$||x| - 1| + ||y| - 1| \leq 1?$$

A. 2 B. 8 C. 4 D. 15 E. 12

Solution: $\boxed{\textbf{(B) } 8}$.

14. (2023 AMC 10B Problem 14)

How many ordered pairs of integers (m, n) satisfy the equation $m^2 + mn + n^2 = m^2n^2$?

A. 7 B. 1 C. 3 D. 6 E. 5

Solution: Case 1: $mn = 0$.

In this case, $m = n = 0$.

Case 2: $mn \neq 0$.

Denote $k = \gcd(m, n)$. Denote $m = ku$ and $n = kv$. Thus, $\gcd(u, v) = 1$.

Thus, the equation given in this problem can be written as

$$u^2 + uv + v^2 = k^2u^2v^2.$$

Modulo u, we have $v^2 \equiv 0 \pmod{u}$. Because $(u, v) = 1$, we must have $|u| = |v| = 1$. Plugging this into the above equation, we get $2 + uv = k^2$. Thus, we must have $uv = -1$ and $k = 1$.

Thus, there are two solutions in this case: $(m, n) = (1, -1)$ and $(m, n) = (-1, 1)$.

Putting all cases together, the total number of solutions is **(C) 3**.

15. (2023 AMC 10B Problem 15)

What is the least positive integer m such that $m \cdot 2! \cdot 3! \cdot 4! \cdot 5! \cdots 16!$ is a perfect square?

A. 30 B. 30,030 C. 70 D. 1430 E. 1001

Solution: We have

$$\begin{aligned} 2! \cdot 3! \cdot 4! \cdot 5! \cdots 16! &= 2^{15}3^{14}4^{13} \cdots 15^2 16 \\ &= \left(2^{14}3^{14}4^{12}5^{12} \cdots 12^4 13^4 14^2 15^2\right) \\ &\quad \cdot 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16 \\ &= \left(2^{14}3^{14}4^{12}5^{12} \cdots 12^4 13^4 14^2 15^2\right) \\ &\quad \cdot (2 \cdot 4 \cdot 8 \cdot 16) \cdot (6 \cdot 10 \cdot 12 \cdot 14) \\ &= \left(2^{14}3^{14}4^{12}5^{12} \cdots 12^4 13^4 14^2 15^2 16\right) \\ &\quad \cdot (2 \cdot 4 \cdot 8 \cdot 16) \cdot \left(6^2 \cdot 2^2\right) \cdot 5 \cdot 14. \end{aligned}$$

Therefore, the smallest m is $5 \cdot 14 =$ **(C) 70**.

16. (2023 AMC 10B Problem 16)

Define an *upno* to be a positive integer of 2 or more digits where the digits are strictly increasing moving left to right. Similarly, define a *downno* to be a positive integer of 2 or more digits where the digits are strictly decreasing moving left to right. For instance, the number 258 is an upno and 8620 is a downno. Let U equal the total number of upnos and D equal the total number of downnos. What is $|U - D|$?

A. 512 B. 10 C. 0 D. 9 E. 511

Solution: We establish a mapping from the upno set to the downno set by reversing all digits. Thus, each number on the upno set is mapped to a number in the downno set.

Under this mapping function, all numbers in the downno set that cannot be inversely mapped from the upno set all end with 0. Denote by i the number of digits in a downno number that ends with 0. Therefore,

$$\begin{aligned} |U-D| &= D-U \\ &= \sum_{i=1}^{9} \binom{9}{i} \\ &= \sum_{i=0}^{9} \binom{9}{i} - \binom{9}{0} \\ &= 2^9 - 1 \\ &= \boxed{\textbf{(E) 511}}. \end{aligned}$$

17. (2023 AMC 10B Problem 17)(2023 AMC 12B Problem 13)

A rectangular box P has distinct edge lengths a, b, and c. The sum of the lengths of all 12 edges of P is 13, the sum of the areas of all 6 faces of P is $\frac{11}{2}$, and the volume of P is $\frac{1}{2}$. What is the length of the longest interior diagonal connecting two vertices of P?

A. 2 B. $\frac{3}{8}$ C. $\frac{9}{8}$ D. $\frac{9}{4}$ E. $\frac{3}{2}$

Solution: $\boxed{\textbf{(D) } \frac{9}{4}}$.

18. (2023 AMC 10B Problem 18)(2023 AMC 12B Problem 15)

Suppose that a, b, and c are positive integers such that

$$\frac{a}{14} + \frac{b}{15} = \frac{c}{210}.$$

Which of the following statements are necessarily true?

1. If $\gcd(a, 14) = 1$ or $\gcd(b, 15) = 1$ or both, then $\gcd(c, 210) = 1$.
2. If $\gcd(c, 210) = 1$, then $\gcd(a, 14) = 1$ or $\gcd(b, 15) = 1$ or both.
3. $\gcd(c, 210) = 1$ if and only if $\gcd(a, 14) = 1$ and $\gcd(b, 15) = 1$.

A. I, II, and III B. I only C. I and II only D. III only E. II and III only

Solution: $\boxed{\textbf{(E) II and III only}}$.

19. (2023 AMC 10B Problem 19)

Sonya the frog chooses a point uniformly at random lying within the square $[0,6]\times[0,6]$ in the coordinate plane and hops to that point. She then chooses a distance uniformly at random in the interval $[0,1]$ and a direction uniformly at random from {north, east, south, west}. All her choices are independent. She now hops the chosen distance in the chosen direction. What is the probability that she lands outside the square?

A. $\frac{1}{6}$ B. $\frac{1}{12}$ C. $\frac{1}{4}$ D. $\frac{1}{10}$ E. $\frac{1}{9}$

Solution: We denote by (x,y) the frog's initial coordinates. We denote by $k\in\{n,e,s,w\}$ the direction to hop. We denote by z the hopping distance. In this analysis, we say that the frog wins if landing outside the square.

We have

$$\begin{aligned}
P(win) &= \sum_{k\in\{n,e,s,w\}} P(win|k)\,P(k) \\
&= P(win|k=w)\sum_{k\in\{n,e,s,w\}} P(k) \\
&= P(win|k=w) \\
&= \int_{y=0}^{1} P(win|k=w,y)\,dy \\
&= P(win|k=w,y=0) \\
&= P(win|k=w,y=0,x\in[0,1])\,P(x\in[0,1]) + P(win|k=w,y=0,x\in(1,6])\,P(x\in(1,6]) \\
&= P(win|k=w,y=0,x\in[0,1])\cdot\frac{1}{6} + 0\cdot\frac{5}{6} \\
&= \frac{1}{6}P(win|k=w,y=0,x\in[0,1]) \\
&= \frac{1}{6}\int_{x=0}^{1}\int_{z=x}^{1} dz\,dx \\
&= \frac{1}{6}\cdot\frac{1}{2} \\
&= \boxed{\textbf{(B) } \frac{1}{12}}.
\end{aligned}$$

The second equality above follows from symmetry that $P(win|k)$ are all the same for all $k\in\{n,e,s,w\}$. The fifth equality above follows from symmetry that $P(win|k=w,y)$ are all the same for all $y\in[0,1]$.

20. (2023 AMC 10B Problem 20)

Four congruent semicircles are drawn on the surface of a sphere with radius 2, as shown, creating a closed curve that divides its surface into two congruent regions. The length of the curve is $\pi\sqrt{n}$. What is n?

A. 32 B. 12 C. 48 D. 36 E. 27

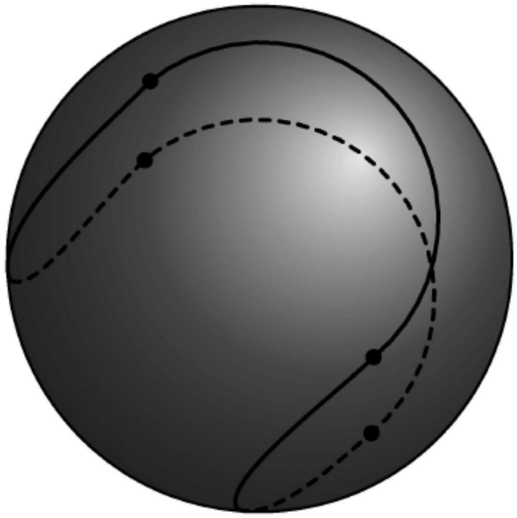

Figure 2: 2023 AMC 10B Problem 20

Solution: We put the sphere to a coordinate space by putting the center at the origin. The four connecting points of the curve have the following coordinates: $A = (0, 0, 2)$, $B = (2, 0, 0)$, $C = (0, 0, -2)$, $D = (-2, 0, 0)$.

Now, we compute the radius of each semicircle. Denote by M the midpoint of A and B. Thus, M is the center of the semicircle that ends at A and B. We have $M = (1, 0, 1)$. Thus, $OM = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$.

In the right triangle $\triangle OAM$, we have $MA = \sqrt{OA^2 - OM^2} = \sqrt{2}$.

Therefore, the length of the curve is

$$4 \cdot \frac{1}{2} 2\pi \cdot MA = \pi\sqrt{32}.$$

Therefore, the answer is $\boxed{\textbf{(A) } 32}$.

21. (2023 AMC 10B Problem 21)(2023 AMC 12B Problem 19)
Each of 2023 balls is randomly placed into one of 3 bins. Which of the following is closest to the probability that each of the bins will contain an odd number of balls?
A. $\frac{2}{3}$ B. $\frac{3}{10}$ C. $\frac{1}{2}$ D. $\frac{1}{3}$ E. $\frac{1}{4}$

Solution: $\boxed{\textbf{(E) } \frac{1}{4}}$.

22. (2023 AMC 10B Problem 22)

How many distinct values of x satisfy $\lfloor x \rfloor^2 - 3x + 2 = 0$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x?

A. an infinite number B. 4 C. 2 D. 3 E. 0

Solution: Denote $a = \lfloor x \rfloor$. Denote $b = x - \lfloor x \rfloor$. Thus, $b \in [0, 1)$.

The equation given in this problem can be written as

$$a^2 - 3(a + b) + 2 = 0.$$

Thus,

$$3b = a^2 - 3a + 2.$$

Because $b \in [0, 1)$, we have $3b \in [0, 3)$. Thus,

$$a^2 - 3a + 2 = 0, 1, \text{ or } 2.$$

Therefore, $a = 1$, 2, 0, 3. Therefore, the number of solutions is **(B) 4**.

23. (2023 AMC 10B Problem 23)

An arithmetic sequence of positive integers has $n \geq 3$ terms, initial term a, and common difference $d > l$. Carl wrote down all the terms in this sequence correctly except for one term, which was off by 1. The sum of the terms he wrote down was 222. What is $a + d + n$?

A. 24 B. 20 C. 22 D. 28 E. 26

Solution: The total sum is

$$\frac{(a + (a + d(n-1)))\, n}{2} = 221 \text{ or } 223.$$

Thus,

$$(2a + d(n-1))\, n = 442 \text{ or } 446.$$

Because $a \in \mathbb{N}$, $d \geq 2$, $n \geq 3$, we have $2a + d(n-1) \geq 6$.

The prime factorization of 446 is $2 \cdot 223$. So there is no feasible solution.

The prime factorization of 442 is $2 \cdot 13 \cdot 17$. Hence, the unique feasible solution is $a = 5$, $d = 2$, $n = 13$. Therefore, $a + d + n =$ **(B) 20**.

24. (2023 AMC 10B Problem 24)

What is the length of the boundary of the region in the xy plane consisting of points of the form $(2u - 3w, v + 4w)$ where $0 \le u \le 1$, $0 \le v \le 1$, and $0 \le w \le 1$?

A. $10\sqrt{3}$ B. 13 C. 15 D. 18 E. 16

Solution: The region is polygon with the following vertices: $(0,0)$, $(2,0)$, $(2,1)$, $(-1,5)$, $(-3,5)$, $(-3,4)$.

Therefore, the length of the boundary is $2+1+5+2+1+5 =$ **(E) 16**.

25. (2023 AMC 10B Problem 25)(2023 AMC 12B Problem 25)

A regular pentagon with area $1+\sqrt{5}$ is printed on paper and cut out. All five vertices are folded to the center of the pentagon, creating a smaller pentagon. What is the area of the new pentagon?

A. $4-\sqrt{5}$ B. $\sqrt{5}-1$ C. $8-3\sqrt{5}$ D. $\frac{1+\sqrt{5}}{2}$ E. $\frac{2+\sqrt{5}}{3}$

Solution: **(B)** $\sqrt{5}-1$.